

Banach Space X-Valued Bilateral Sequence Space

$\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$

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Abstract

Aims/ objectives:

In this paper we introduce and study vector-valued bilateral sequence space $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. We investigate the conditions connected with the comparison of the classes in terms of $\bar{\lambda}$ and $\bar{p}$ so that a class is contained in or equal to another class of same kind. We also study topological linear structure of this space when this space is topologized by a suitable paranorm.

Keywords: Bilateral Sequence, Sequence Space and Paranormed space

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1 INTRODUCTION

So far, a good number of research works have been done on bilateral sequence spaces for instance, see [2], [7], [3], [12] and [13]. Leon and Montes, in [9] has worked on the complex bilateral sequence space $\ell^2(\mathbb{Z})$ to obtain various results on hypercyclic bilateral weighted shift. In [8], Menet generalized this result to the complex bilateral sequence spaces $\ell^p(\mathbb{Z})$ with $1 \leq p < \infty$ and $c_0(\mathbb{Z})$ and afterwards, to the complex weighted spaces $\ell^p(v, \mathbb{Z})$ and $c_0(v, \mathbb{Z})$. Shkarin, in [11] and [10] used the bilateral sequence spaces $\ell_\infty(\mathbb{Z})$, $\ell_p(\mathbb{Z})$ with $1 \leq p < \infty$ and $c_0(\mathbb{Z})$ to obtain various results associated with weighted bilateral shift on these spaces and also used $\{f_j\}_{j \in \mathbb{Z}}$, a sequence of elements of $\mathcal{B}$ where $\mathcal{B}$ is a Banach space. We have introduced and studied the Banach space X -valued bilateral sequence spaces $c_0(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$, $c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ in [15] involving bilateral sequence $\bar{p} = (p_k)_{-\infty}^{\infty}$ and the multiplier $\bar{\lambda} = (\lambda_k)_{-\infty}^{\infty}$.



By a bilateral sequence we mean a function whose domain is the set $\mathbb{Z}$ of all integers with natural ordering. We will denote a bilateral sequence by the symbol $(a_k)_{-\infty}^{\infty}$ or $\bar{a} = (a_k)_{-\infty}^{\infty}$. As usual by the convergence of the bilateral series $\sum_{-\infty}^{\infty} a_k$ to s written as $\sum_{-\infty}^{\infty} a_k = s$ we shall mean the convergence

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of the sequence $(s_n)_{n=1}^{\infty}$ to s . Let $\bar{p} = (p_k)_{k=-\infty}^{\infty}$ and $\bar{q} = (q_k)_{k=-\infty}^{\infty}$ be bilateral sequences of strictly positive real numbers and $\bar{\lambda} = (\lambda_k)_{k=-\infty}^{\infty}$ and $\bar{\mu} = (\mu_k)_{k=-\infty}^{\infty}$ be bilateral sequences of non-zero complex numbers. Our aim in this paper is to investigate the results concerning the class $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ defined below as a generalization of the sequence space $\ell(X, \bar{\lambda}, \bar{p})$ (studied by Srivastava and Srivastava [16]) which is itself generalization of well known complex sequence space $\ell(\bar{p})$, $\bar{p} = (p_k)_{k=1}^{\infty}$ studied by Maddox [6] and many others, for instance, see [13], [5], [14] and [1]. We define

$$\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) = \left\{ \bar{x} = (x_k)_{k=-\infty}^{\infty} : x_k \in X, k \in \mathbb{Z}, \left(\sum_{k=-\infty}^{\infty} \|\lambda_k x_k\|^{p_k} \right) < \infty \right\}.$$

If $p_k = 1$ for all $k \in \mathbb{Z}$ in $\bar{p} = (p_k)_{k=-\infty}^{\infty}$, we shall denote $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ by $\ell(\mathbb{Z}, X, \bar{\lambda})$ and if $\lambda_k = 1$ for all $k \in \mathbb{Z}$ in $\bar{\lambda} = (\lambda_k)_{k=-\infty}^{\infty}$ then we shall denote $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ by $\ell(\mathbb{Z}, X, \bar{p})$.

We shall also need

$$\ell_{\infty}(\mathbb{Z}, \mathbb{R}) = \{ \bar{a} = (a_k)_{k=-\infty}^{\infty} : a_k \in \mathbb{R}, k \in \mathbb{Z}, \sup_k |a_k| < \infty \}$$

Throughout the paper, for each $k \in \mathbb{Z}$, we shall denote $t_k = \left| \frac{\lambda_k}{\mu_k} \right|^{p_k}$ and $M = \max(1, \sup_k p_k)$ and shall denote by $\mathbb{Z}(m, n)$ the open integer interval defined as

$$z(m, n) = \begin{cases} m+1, m+2, \dots, n-2, n-1, & m+1 \leq n-1 \\ \phi, & \text{otherwise} \end{cases}$$

Also we shall denote complement of $\mathbb{Z}(m, n)$ by $\mathbb{Z} \setminus \mathbb{Z}(m, n)$.

Definition 1.1. Let X be a linear space. A mapping $g : X \rightarrow \mathbb{R}$ is called a paranorm if it satisfies following conditions :

- (i) $g(\theta) = 0$
- (ii) $g(x) = g(-x)$
- (iii) $g(x+y) \leq g(x) + g(y)$
- (iv) if (α_n) is a sequence of scalars with $\alpha_n \rightarrow \alpha$ and (x_n) is a sequence in X with $g(x_n - x) \rightarrow 0$ then $g(\alpha_n x_n - \alpha x) \rightarrow 0$ (continuity of scalar multiplication).

The paranorm is called total if

- (v) $g(x) = 0$ implies $x = 0$, see [17].

2 CONTAINMENT

In this section conditions for containment relations of $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ in terms of $\bar{p}$ and $\bar{\lambda}$ are investigated.

Lemma 2.1. $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{\mu}, \bar{p})$ if and only if

$$\lim_{k \rightarrow -\infty} \inf t_k > 0 \text{ and } \lim_{k \rightarrow \infty} \inf t_k > 0$$

Proof. Suppose $\lim_{k \rightarrow -\infty} \inf t_k > 0$ and $\lim_{k \rightarrow \infty} \inf t_k > 0$ and $\bar{x} = (x_k)_{k=-\infty}^{\infty} \in \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. Then there exists $m > 0$ such that $m|\mu_k|^{p_k} < |\lambda_k|^{p_k}$, for all sufficiently large values of $|k|$. Thus $m\|\mu_k x_k\|^{p_k} < \|\lambda_k x_k\|^{p_k}$, for all sufficiently large values of $|k|$. Now we easily get that $\bar{x} \in \ell(\mathbb{Z}, X, \bar{\mu}, \bar{p})$. Hence $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{\mu}, \bar{p})$.

Conversely suppose that $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{\mu}, \bar{p})$ but $\lim_{k \rightarrow -\infty} \inf t_k = 0$ and/or $\lim_{k \rightarrow \infty} \inf t_k = 0$. Let us take the case when only $\lim_{k \rightarrow \infty} \inf t_k = 0$. Now we can find a sequence of integers $(k(n))$ such that $1 \leq k(n) < k(n+1)$, $n \geq 1$ for which $n|\lambda_{k(n)}|^{p_{k(n)}} < |\mu_{k(n)}|^{p_{k(n)}}$. We now see that $\bar{x} = (x_k)_{k=-\infty}^{\infty}$ defined by

$$x_k = \begin{cases} \lambda_{k(n)}^{-1} n^{-2/p_{k(n)}} z, & \text{if } k = k(n), n \geq 1 \text{ and} \\ \theta, & \text{otherwise} \end{cases}$$

where $z \in X$, and $\|z\| = 1$, is in $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ but not in $\ell(\mathbb{Z}, X, \bar{\mu}, \bar{p})$ as

$$\sum_{k=-\infty}^{\infty} \|\lambda_k x_k\|^{p_k} < \infty \text{ and} \\ \sum_{k=-\infty}^{\infty} \|\mu_k x_k\|^{p_k} = \sum_{n=1}^{\infty} \left| \frac{\mu_{k(n)}}{\lambda_{k(n)}} \right|^{p_{k(n)}} \cdot \frac{1}{n^2} > \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot n = \infty,$$

a contradiction to our assumption.

Similar proof can be given for the other cases. This completes the proof. □

Lemma 2.2. $\ell(\mathbb{Z}, X, \bar{\mu}, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ if and only if

$$\lim_{k \rightarrow -\infty} \sup t_k < \infty \text{ and } \lim_{k \rightarrow \infty} \sup t_k < \infty$$

Proof. Sufficiency of the condition can easily be proved on the lines of above **Lemma 2.1**. For the necessity let us suppose that $\ell(\mathbb{Z}, X, \bar{\mu}, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ but $\lim_{k \rightarrow -\infty} \sup t_k = \infty$ and/or $\lim_{k \rightarrow \infty} \sup t_k = \infty$. We take the case when only $\lim_{k \rightarrow \infty} \sup t_k = \infty$. Then there exists a sequence of integers $(k(n))$ such that, $1 \leq k(n) < k(n+1)$, $n \geq 1$ for which

$$|\lambda_{k(n)}|^{p_{k(n)}} > n|\mu_{k(n)}|^{p_{k(n)}}.$$

Let $z \in X$ with $\|z\| = 1$ and consider the sequence $\bar{x} = (x_k)_{k=-\infty}^{\infty}$ defined in the proof of Lemma 2.1. We easily see that $\bar{x} = (x_k)_{k=-\infty}^{\infty}$ is in $\ell(\mathbb{Z}, X, \bar{\mu}, \bar{p})$ but $\bar{x} \notin \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ as

$$\sum_{k=-\infty}^{\infty} \|\lambda_k x_k\|^{p_k} = \sum_{n=1}^{\infty} \left| \frac{\lambda_{k(n)}}{\mu_{k(n)}} \right|^{p_{k(n)}} \cdot \frac{1}{n^2} > \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot n = \infty,$$

a contradiction to our assumption that $\ell(\mathbb{Z}, X, \bar{\mu}, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$.

This completes the proof. □



When Lemmas 2.1 and 2.2 are combined, we get:

Theorem 2.3. $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) = \ell(\mathbb{Z}, X, \bar{\mu}, \bar{p})$ if and only if

$$0 < \lim_{k \rightarrow -\infty} \inf t_k < \lim_{k \rightarrow -\infty} \sup t_k < \infty \text{ and} \\ 0 < \lim_{k \rightarrow \infty} \inf t_k < \lim_{k \rightarrow \infty} \sup t_k < \infty.$$

Corollary 2.4. (i) $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{p})$ if and only if

$$\lim_{k \rightarrow -\infty} \inf |\lambda_k|^{p_k} > 0 \text{ and } \lim_{k \rightarrow \infty} \inf |\lambda_k|^{p_k} > 0.$$

(ii) $\ell(\mathbb{Z}, X, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ if and only if

$$\lim_{k \rightarrow -\infty} \sup |\lambda_k|^{p_k} < \infty \text{ and } \lim_{k \rightarrow \infty} \sup |\lambda_k|^{p_k} < \infty.$$

(iii) $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) = \ell(\mathbb{Z}, X, \bar{p})$ if and only if

$$0 < \lim_{k \rightarrow -\infty} \inf |\lambda_k|^{p_k} < \lim_{k \rightarrow -\infty} \sup |\lambda_k|^{p_k} < \infty \text{ and } \\ 0 < \lim_{k \rightarrow \infty} \inf |\lambda_k|^{p_k} < \lim_{k \rightarrow \infty} \sup |\lambda_k|^{p_k} < \infty.$$

Proof. Proof easily follows from Lemma 2.1, Lemma 2.2 and Theorem 2.3. □

Lemma 2.5. If $p_k \leq q_k$ for all but finitely many $k \in \mathbb{Z}$ then

$$\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{q})$$

Proof. Let $p_k \leq q_k$ for all but finitely many $k \in \mathbb{Z}$. If $\bar{x} = (x_k)_{-\infty}^{\infty} \in \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ then clearly $\bar{x} \in \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{q})$ because $||\lambda_k x_k|| \leq 1$ for all large values of $|k|$ and so $||\lambda_k x_k||^{q_k} \leq ||\lambda_k x_k||^{p_k}$, for all large values of $|k|$. This completes the proof. □

Theorem 2.6. If

(i) $\lim_{k \rightarrow -\infty} \inf t_k > 0$ and $\lim_{k \rightarrow \infty} \inf t_k > 0$, and

(ii) $p_k \leq q_k$, for all but finitely many $k \in \mathbb{Z}$, then

$$\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{\mu}, \bar{q})$$

Proof. Proof easily follows from Lemmas 2.1 and 2.5. □

3 PARANORMED SPACE STRUCTURE OF $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$

As usual for the sequence spaces, here also the linear space structure of $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ over the field $\mathbb{C}$ of complex numbers is concerned, vector operations will be taken co-ordinatewise i.e., $\bar{x} + \bar{y} = (x_k + y_k)_{-\infty}^{\infty}$ and $\alpha \bar{x} = (\alpha x_k)_{-\infty}^{\infty}$. Further we note that $\bar{p} = (p_k)_{-\infty}^{\infty} \in \ell_{\infty}(\mathbb{Z}, \mathbb{R})$ is a necessary and sufficient condition for the linearity of $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. Therefore throughout the section we shall take $\bar{p} = (p_k)_{-\infty}^{\infty} \in \ell_{\infty}(\mathbb{Z}, \mathbb{R})$.

We define

$$(3.1) \quad Q_{\bar{\lambda}, \bar{p}}(\bar{x}) = \left(\sum_{-\infty}^{\infty} ||\lambda_k x_k||^{p_k} \right)^{1/M}$$

for $\bar{x} \in \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ where $M = \max(1, \sup_k p_k)$.

recall the following definition (see [15]):

Definition 3.1. Let $E(X)$ be the linear space of normed space X -valued bilateral sequences $\bar{x} = (x_k)_{-\infty}^{\infty}$ and $x \in X$. We define

(i) $\delta_n(x) = (\dots, \theta, x, \theta, \dots)$, where x is at n th place, $n \in \mathbb{Z}$.

(ii) $E(X)$ equipped with the linear topology $\mathcal{T}$ is said to be a GK -space if the map $P_n : E(X) \rightarrow X, P_n(\bar{x}) = x_n$, is continuous for each $n \in \mathbb{Z}$.

A GK -space is called

(iii) a GAD -space if $\Phi(X)$ is dense in $E(X)$, where

$\Phi(X) = \{\bar{x} = (x_k)_{-\infty}^{\infty} : x_k \in X, k \in \mathbb{Z} \text{ and } x_k = \theta, \text{ for all but finitely many } k\}$,

(iv) a GAk -space if for each $\bar{x} = (x_k)_{-\infty}^{\infty}$ in $E(X)$, $s_n(\bar{x}) \rightarrow \bar{x}$ as $n \rightarrow \infty$ with respect to $\mathcal{T}$, where $s_n(\bar{x}) = (\dots, \theta, x_{-n}, \dots, x_{-1}, x_0, x_1, \dots, x_n, \theta, \dots)$,

(v) a GFK -space if $(E(X), \mathcal{T})$ is complete linear metric space,

(vi) a GC -space if $R_n : X \rightarrow E(X), R_n(x) = \delta_n(x)$ is continuous for each $n \in \mathbb{Z}$.

Where GK -space, GAk -space, GFK -space and GC -space are generalized versions defined for vector valued bilateral sequences corresponding to K -space, AK -space, FK -space and C -space which are defined for scalar sequences (see Wilansky [17] and Kamthan and Gupta [4]).

Theorem 3.1. Let X be a normed space and consider $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ with $Q_{\bar{\lambda}, \bar{p}}$.

(i) $(\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}), Q_{\bar{\lambda}, \bar{p}})$ is a total paranormed GK -space,

(ii) $(\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}), Q_{\bar{\lambda}, \bar{p}})$ is a GAD -, GAk - and GC -space,

(iii) if X is separable then so is $(\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}), Q_{\bar{\lambda}, \bar{p}})$ and

(iv) if X is a Banach space then $(\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}), Q_{\bar{\lambda}, \bar{p}})$ is a GFK -space.

Proof. Throughout the theorem $Q_{\bar{\lambda}, \bar{p}}$ will be denoted by Q

(i) We prove here continuity of scalar multiplication only and other conditions for Q to be a total paranorm can be proved very easily. To prove the continuity of scalar multiplication, it is sufficient to show:

(a) $Q(\bar{x}^{(n)}) \rightarrow 0$ and $\alpha_n \rightarrow \alpha$ imply $Q(\alpha_n \bar{x}^{(n)}) \rightarrow 0$, and

(b) $\alpha_n \rightarrow 0$ imply $Q(\alpha_n \bar{x}) \rightarrow 0$ for each $\bar{x} \in \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$

Let $Q(\bar{x}^{(n)}) = \left(\sum_{-\infty}^{\infty} \|\lambda_k x_k^{(n)}\|^{p_k} \right)^{1/M} \rightarrow 0$ and $\alpha_n \rightarrow \alpha$ as $n \rightarrow \infty$.

Further suppose that for some $L > 0$ $|\alpha_n| \leq L$, for all $n \geq 1$.

$$\begin{aligned} \text{Then } Q(\alpha_n \bar{x}^{(n)}) &= \left(\sum_{-\infty}^{\infty} \|\lambda_k \alpha_n x_k^{(n)}\|^{p_k} \right)^{1/M} \\ &= \left(\sum_{-\infty}^{\infty} |\alpha_n|^{p_k} \|\lambda_k x_k^{(n)}\|^{p_k} \right)^{1/M} \\ &\leq \left(\sum_{-\infty}^{\infty} L^{p_k} \|\lambda_k x_k^{(n)}\|^{p_k} \right)^{1/M} \\ &= \sup(L^{p_k/M}) \left(\sum_{-\infty}^{\infty} \|\lambda_k x_k^{(n)}\|^{p_k} \right)^{1/M} \\ &\leq A(L) Q(\bar{x}^{(n)}) \end{aligned}$$

where $A(L) = \max(1, L)$. This implies that $Q(\alpha_n \bar{x}^{(n)}) \rightarrow 0$ as $n \rightarrow \infty$. This proves (a).

Now let $\bar{x} \in \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ then for $\epsilon > 0$ there exists K such that

$$\sum_{k \in \mathbb{Z} \setminus \mathbb{Z}(-K, K)} \|\lambda_k x_k^{(n)}\|^{p_k} < \left(\frac{\epsilon}{2}\right)^M.$$

Further if $\alpha_n \rightarrow 0$ we can find N such that for $n \geq N$ and $|\alpha_n| \leq 1$

$$\sum_{k \in \mathbb{Z}(-K, K)} |\alpha_n|^{p_k} \|\lambda_k x_k\|^{p_k} < \left(\frac{\epsilon}{2}\right)^M \text{ and } |\alpha_n| \leq 1.$$

Thus, for all $n \geq N$ we get

$$Q(\alpha_n \bar{x}) \leq \left(\sum_{k \in \mathbb{Z}(-K, K)} \|\alpha_n \lambda_k x_k\|^{p_k} \right)^{1/M} + \left(\sum_{k \in \mathbb{Z} \setminus \mathbb{Z}(-K, K)} \|\lambda_k x_k\|^{p_k} \right)^{1/M} \\ \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and hence (b) follows.

Further each $k \in \mathbb{Z}$, the continuity of $P_k : \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) \rightarrow X$ where $P_k(\bar{x}) = x_k$ follows from

$$P_k(\bar{x}) = \|x_k\| \leq |\lambda_k|^{-1} (Q(\bar{x}))^{M/p_k}.$$

Thus $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ is a GK -space which proves (i).

(ii) Let $\bar{x} = (x_k) \in \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ and $\epsilon > 0$. Then there exists K such that

$$\left(\sum_{k \in \mathbb{Z} \setminus \mathbb{Z}(-K, K)} \|\lambda_k x_k\|^{p_k} \right)^{1/M} < \epsilon$$

We now easily see that

$$Q(\bar{x} - s_{K-1}(\bar{x})) = \left(\sum_{k \in \mathbb{Z} \setminus \mathbb{Z}(-K, K)} \|\lambda_k x_k\|^{p_k} \right)^{1/M} < \epsilon.$$

This shows that $\Phi(X)$ is dense in $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ as well as $s_n(\bar{x}) \rightarrow \bar{x}$ as $n \rightarrow \infty$. Hence $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ is a GAD -space and also a GAK -space.

Now let $R_k : X \rightarrow \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$, where $R_k(x) = \delta_k(x)$, $k \in \mathbb{Z}$. Clearly continuity of $R_k, k \in \mathbb{Z}$ follows from

$$Q(R_k(x)) = Q(\delta_k(x)) \leq |\lambda_k|^{p_k/M} \|x\|^{p_k/M}.$$

Hence $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ is a GC -space.

(iii) If D is a countable dense subset of X then $\Phi(D)$ will be a countable dense subset of $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$.

(iv) If X is a Banach space then we show that $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ is complete with respect to the metric induced by Q .

Let $(\bar{x}^{(n)})$ be a Cauchy sequence in $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. Then for $\epsilon > 0$ there exists N such that

$$(3.2) \quad \left(\sum_{k=-\infty}^{\infty} \|\lambda_k x_k^{(n)} - \lambda_k x_k^{(m)}\|^{p_k} \right)^{1/M} < \epsilon, \quad \text{for all } n, m \geq N,$$

and so for each $k \in \mathbb{Z}$

$$\|x_k^{(n)} - x_k^{(m)}\| < |\lambda_k|^{-1} \epsilon^{M/p_k} < |\lambda_k|^{-1} \epsilon, \quad \text{for all } n, m \geq N.$$

This shows that for each $k \in \mathbb{Z}$, $(x_k^{(n)})_{n=1}^{\infty}$ is a Cauchy sequence in X . Since X is complete therefore $x_k^{(n)} \rightarrow x_k \in X$ as $n \rightarrow \infty$. Let $\bar{x} = (x_k) \in X$. Since $(\bar{x}^{(n)})$ is a Cauchy sequence therefore it will be bounded with respect to Q . Suppose

$$Q(\bar{x}^{(n)}) = \left(\sum_{k=-\infty}^{\infty} \|\lambda_k x_k^{(n)}\|^{p_k} \right)^{1/M} \leq L,$$

for some $L > 0$ and each $n \geq 1$. Now for any $t \geq 1$, we have

$$\left(\sum_{-t}^t \|\lambda_k x_k^{(n)}\|^{p_k} \right)^{1/M} \leq \left(\sum_{-\infty}^{\infty} \|\lambda_k x_k^{(n)}\|^{p_k} \right)^{1/M} \leq L$$

and so taking $n \rightarrow \infty$ then $t \rightarrow \infty$ we get

$$\left(\sum_{-\infty}^{\infty} \|\lambda_k x_k\|^{p_k} \right)^{1/M} \leq L.$$

This shows that $\bar{x} \in \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$.

Now it remains to show that $Q(\bar{x}^{(n)} - \bar{x}) \rightarrow 0$ as $n \rightarrow \infty$. By (3.2), for each fixed t we have

$$\sum_{-t}^t \|\lambda_k x_k^{(n)} - \lambda_k x_k^{(m)}\|^{p_k} < \epsilon^M.$$

Thus if in this inequality we take $m \rightarrow \infty$ first and then $t \rightarrow \infty$ we easily get

$$\left(\sum_{-\infty}^{\infty} \|\lambda_k x_k^{(n)} - \lambda_k x_k\|^{p_k} \right)^{1/M} \leq \epsilon, \quad \text{for each } n \geq N$$

and so $Q(\bar{x}^{(n)} - \bar{x}) \rightarrow 0$ as $n \rightarrow \infty$ i.e., $\bar{x}^{(n)} \rightarrow \bar{x}$ with respect to Q .

This proves the completeness of $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. Moreover $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ is a GFK -space because $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ is complete as well as a GK -space. □



References

- [1] BERBERIAN, S. K. **Lectures in Functional Analysis and Operator Theory**. Springer-Ver. New York/Berlin, page 166.
- [2] CHEN, C. (2012). **Supercyclic and Cesàro Hypercyclic Weighted Translations on Groups**. Taiwanese J. Math., **16(5)**:1815–1827.
- [3] DONOHO, D. L AND JOHNSTONE, I. M. (1998). **Minimax Estimation via Wavelet Shrinkage**. The Annals Stat., **26(3)**:879–921.
- [4] KAMTHAN, P. K AND GUPTA, M. (1981). **Sequence Spaces and Series**. Marcel Dekker Inc, New York.
- [5] LASCARIDES, C. G. (1973). **A Study of Certain Sequence Spaces of Maddox and a Generalization of a Theorem of Iyer**. Pacific, J. Math, **38(2)**:487–500.
- [6] MADDOX, I. J. (1980). **Infinite Matrices of Operators (Lecture Notes in Mathematics) 786**. Springer-Ver. Berlin-Heidelberg/New York.
- [7] MATSUZAKI, K. (2010). **An Averaging Operator and Non-separability of Certain Banach Spaces of Holomorphic Automorphic Forms**. RIMS Kôky. Bess., **17**:65–72.
- [8] MENET, Q. (2012). **Hypercyclic Subspaces and Weighted Shifts**. Cornell Univ. Lib. arXiv, **1208.4963**. [Preprint].
- [9] SAAVEDRA, F. L AND RODRIGUEZ, A. M. (2000). **Spectral Theory and Hypercyclic Subspaces**. Trans. Amer. Math. Soc., **353**:247–267.
- [10] SHKARIN, S. (2012). **Non-sequential Weak Supercyclicity and Hypercyclicity**. Cornell Univ. Lib. arXiv, **1209.1462**. [Preprint].
- [11] SHKARIN, S. (2012). **A Weighted Bilateral Shift with Cyclic Square is Supercyclic**. Cornell Univ. Lib. arXiv, **1209.1458**. [Preprint].

- [12] SIKK , J. (1994). **The rate spaces $m(\lambda)$, $c(\lambda)$, $c_0(\lambda)$ and $l^p(\lambda)$ of sequences.** Acta et Comm. Univ. Tart. De Math., **970**:87–96.
- [13] SIMON , N. C AND MARKO , L. W. (2011). **Limit Operators, Collective Compactness, and the Spectral Theory of Infinite Matrices.** Mem. Amer. Math. Soc., **210**.
- [14] SIMONS , S. (1965). **The Sequence Spaces $l(p_\gamma)$ and $m(p_\gamma)$ associated with Schauder Bases.** Proc. London Math. Soc., **15(3)**:422–436.
- [15] SRIVASTAVA , J. K AND AGRAWAL , R. (2012). **Banach Space X-Valued Bilateral Sequence Spaces $c_0(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ and $c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$.** British J. Math. Comp. Sc. [Communicated-BJMCS/2163].
- [16] SRIVASTAVA , J. K AND SRIVASTAVA , B. K. (1996). **Generalized Sequence Spaces $c_0(X, \lambda, p)$.** Indian J. Pure Appl. Math., **27(1)**:73–84.
- [17] WILANSKY , A. (1978). **Modern Methods in Topological Vector Spaces.** McGraw-Hill Inc.,US.

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